A NOTE OF CORRECTION TO A THEOREM OF W. E. BONNICE AND R. J. SILVERMAN

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1. **Introduction.** In [1] W. E. Bonnice and R. J. Silverman proved that a finite-dimensional partially ordered vector space V has the Hahn-Banach extension property if and only if V has the least upper bound property. In the proof of their main theorem (Theorem 6.2) there appear two errors. The purpose of this note is to indicate the errors and to supply a modification for the proof of this theorem. In this note, the terminology, notation and results of [1] are assumed.

2. **The mistakes that appear in [1, Theorem 6.2].** In the following, Theorem 6.2 and a part of the original proof are quoted from [1] for easy reference.

"THEOREM 6.2. *If K is a finite-dimensional wedge with the HBEP then K is lineally closed*" (line 34, p. 218).

"**Proof of 6.2.** By Theorem 4.1, if the finite-dimensional wedge K has the HBEP then $\bar{K}$ has the LUBP. Further, K can be assumed to be a proper reproducing wedge in its containing OLS V, and can be assumed to have nonvoid interior. Moreover, by Theorem 3.1, there is a k-partly positive basis B of dimension n for $\bar{K}$, where $1 \leqq k \leqq n$.

Let $T = \{v \mid -v, v \in \bar{K}\}$. It is easily verified that T is a closed linear subspace of V. Assume $K \neq \bar{K}$ and K has the HBEP" (lines 11–16, p. 219).

"*Case* 1. $T = \{0\}$." (The original proof is correct.)

"*Case* 2. $T \neq \{0\}$." There are several subcases.

"*Case* (2i). $T \cap K = \{0\}$." (The original proof is correct.)

"*Case* (2ii). $T \subset K$." (The original proof is correct.)

"*Case* (2iii). Suppose $T \cap K$ is not closed. Then *by induction on dimension*, K does not have the HBEP, for $T \cap K$ has dimension less than that of K and it will be shown that: If K has the HBEP, then $T \cap K$ has the HBEP" (lines 28–31, p. 220).

Mistake 1. The proof of Case (2iii) cannot be established by induction on dimension because Case (2iii) is only a particular case and $T \cap K$ is not necessarily a wedge satisfying the condition of Case (2iii). In this case we wish to verify the statement $P(n)$: "If K is an n-dimensional wedge satisfying the condition of Case (2iii) and if K is not closed, then K does not have the HBEP." Unfortunately, the following example shows that the wedge $K' = T \cap K$ is not necessarily a wedge satisfying the condition of Case (2iii), i.e., $K' \cap T'$ may be closed, where $T' =$

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$\{v \in T \mid v, -v \in \bar{K}'\}$. Hence we cannot apply the induction assumption that $P(i)$ is true for all $i < n$ to K'.

Counterexample 1. Let $V = R_3$, the linear space of ordered triples of real numbers, $K = \{(x_1, x_2, x_3) \in R_3 \mid x_1 > 0;$ or $x_1 = 0,\ x_3 > 0;$ or $x_1 = 0,\ x_3 = 0,\ x_2 \geqq 0\}$ and let $B = \{b_1, b_2, b_3\}$ where $b_1 = (1, 0, 0)$, $b_2 = (0, 1, 0)$ and $b_3 = (0, 0, 1)$. Then K is a reproducing wedge of V which is not closed and $\bar{K} = \{(x_1, x_2, x_3) \in R_3 \mid x_1 \geqq 0\}$ has a 1-partly positive basis B. Moreover, $T = \{(0, x_2, x_3) \in R_3\}$,

$$K' = K \cap T = \{(0, x_2, x_3) \in R_3 \mid x_3 > 0; \text{or } x_3 = 0,\ x_2 \geqq 0\}$$

is not closed, $\bar{K}' = \{(0, x_2, x_3) \in R_3 \mid x_3 \geqq 0\}$ and $T' = \{(0, x_2, 0) \in R_3\}$. Thus, $K' \cap T' = \{(0, x_2, 0) \in R_3 \mid x_2 \geqq 0\}$ is closed.

"*Case* (2iv). Suppose $T \cap K$ is closed, $T \not\subset K$, $T \cap K \neq \{0\}$ and for every $b \in B$, if $b \in T \cap K$ then $-b \in T \cap K$" (lines 39–40, p. 220). It is easily verified that if $T \cap K$ is closed and has the property that $b \in T \cap K$ implies $-b \in T \cap K$ for every $b \in B$, then $T' \cap K'$ is also closed and has the same property, where $K' = \text{lin } B' \cap K$ and $T' = \{v \in \text{lin } B' \mid v, -v \in \bar{K}'\}$. The original proof of this case is correct if the preceding fact and the results of Case (2i) and Case (2ii) are used in the induction argument.

"*Case* (2v). Suppose $T \cap K$ is closed, $T \not\subset K$, $T \cap K \neq \{0\}$ and there is a $b \in B$, say b_n, so that $b_n \in T \cap K$ and $-b_n \notin T \cap K$. This case breaks down into 2 subcases.

"If $n = 2$, the only wedge which need be considered is C_4" (lines 12–15, p. 221). The proof of this subcase is correct.

"If $n > 2$, let $B' = \{b \in B \mid b \neq b_1\}$ and let $K' = K \cap \text{lin } B'$. Then clearly, K' has dimension less than K. *Also K' is not closed since $T \not\subset K$ and the dimension n is greater than* 2" (lines 31–33, p. 221).

Mistake 2. The above assertion "K' is not closed since $T \not\subset K$ and the dimension n is greater than 2" is not true because there is a counterexample as follows:

Counterexample 2. Let $V = R_3$, let $K = \{(x_1, x_2, x_3) \in R_3 \mid x_1 \geqq 0, x_2 \geqq 0, x_3 \geqq 0;$ or $x_1 > 0, x_2 \geqq 0, x_3 < 0\}$ and let $B = \{b_1, b_2, b_3\}$ where $b_1 = (1, 0, 0)$, $b_2 = (0, 1, 0)$ and $b_3 = (0, 0, 1)$. Then $\bar{K} = \{(x_1, x_2, x_3) \in R_3 \mid x_1 \geqq 0, x_2 \geqq 0\}$ has a 2-partly positive basis B. Moreover, observing that $b_i \in K$, $-b_i \notin K$, $i = 1, 2, 3$, $-b_3 \in \bar{K}$ and $-b_1$, $-b_2 \notin \bar{K}$, therefore $T = \{\lambda b_3 \mid \lambda \in R\}$. It follows that $T \cap K = \{\lambda b_3 \mid \lambda \geqq 0\}$ is closed, $T \not\subset K$, $T \cap K \neq \{0\}$, $b_3 \in T \cap K$ and $-b_3 \notin T \cap K$. Thus, K satisfies all conditions of Case (2v). Let $B' = \{b_2, b_3\}$ and let $K' = K \cap \text{lin } B'$. Then $K' = \{(0, x_2, x_3) \mid x_2 \geqq 0,\ x_3 \geqq 0\}$ is clearly closed.

Since the proof of subcase $n > 2$ of Case (2v) is supported by the false assertion mentioned above, it is incorrect.

3. **A modification of the proof for Case (2iii) and Case (2v).** We firstly give the proof of the theorem for Case (2v), which is based upon the following lemmas:

LEMMA A. *In a 2-dimensional linear space V_2, let $C_2^{(4)} = \{v \in V_2 \mid v = ab_1 + bb_2,$ where $b > 0$; or $a \geqq 0, b = 0\}$ (the open upper-half plane plus the closed bounding* *ray through b_1), where b_1, b_2, determine an appropriate basis for V_2.*(1) *Then $C_2^{(4)}$ is a wedge which does not have the HBEP.*

The proof of Lemma A is the same as the original proof of subcase $n=2$ of Case (2v) of Theorem 6.2.

LEMMA B. *Let $(W; K)$ be a finite-dimensional ordered linear space such that every 2-dimensional linear subspace of W intersects the positive wedge K in a wedge of type $C_2^{(4)}$. Then K does not have the HBEP.*

The proof of Lemma B was given recently in a paper of W. E. Bonnice and R. J. Silverman [2] (from line 21, p. 847 to line 20, p. 848), in which they asserted that Lemma B is true for any dimensional (finite or infinite) ordered linear space V. Their proof is incorrect in general for infinite-dimensional ordered linear spaces(2); however, the proof is correct if $(W; K)$ is a finite-dimensional OLS.

LEMMA C. *Let $(V; C)$ be a finite-dimensional ordered linear space and let $(W; K)$ be an ordered linear subspace of $(V; C)$ with the induced wedge $K=C\cap W$ such that every 2-dimensional linear subspace of W intersects K in a wedge of type $C_2^{(4)}$. Then C does not have the HBEP.*

Proof. Applying the result of Lemma B and using the same argument used in [2] (line 30, p. 845—line 14, p. 847; and line 20, p. 848—line 13, p. 849), Lemma C is established.

A proof of subcase $n>2$ of Case (2v) *of Theorem* 6.2.

Proof. Let $B'=\{b', b_n\}$ where $b'=b_1+\cdots+b_k$ (if $k=1$, $b'=b_1$), and let $K'=\text{lin } B'\cap K$. Then $K'=\{\lambda b'+\lambda_n b_n \mid \lambda>0; \text{ or } \lambda_n\geqq 0, \lambda=0\}$. In fact, since B is a k-partly positive basis for $\bar{K}$, for every $v\in \bar{K}'\subset \text{lin } B'\cap \bar{K}$, $v=\lambda b'+\lambda_n b_n$ where $\lambda, \lambda_n\in R$, $\lambda\geqq 0$. If $\lambda>0$, $v\in \text{Core } K\cap \text{lin } B'\subset \text{Core } K'$; if $\lambda=0$ and $\lambda_n\geqq 0$, $v=\lambda_n b_n\in K'$; if $\lambda=0$ and $\lambda_n<0$, $v=\lambda_n b_n\in \bar{K}'\backslash K'$. Thus, K' is a wedge of type $C_2^{(4)}$ in the linear space lin B' and by Lemma C the positive wedge K of V does not have the HBEP. This contradiction establishes the theorem for this subcase.

A proof of Case (2iii) *of Theorem* 6.2.

Proof. So far we have proved Theorem 6.2 except for Case (2iii). We can now prove this case by mathematical induction. Let $P(i)$ be the statement that if C is an i-dimensional wedge such that C is not closed and such that $T_c\cap C$ is also not closed where $T_c=\{v\in \text{lin } C \mid v, -v\in \bar{C}\}\neq\{0\}$, then C does not have the HBEP. Then $P(1)$ is true since every 1-dimensional wedge is always closed. Assume that

(1) In [1], $C_2^{(4)}$ is denoted by C_4.

(2) A mistake appears in the proof of [2] (line 26, p. 847) where it is asserted that if $(W; K)$ is an OLS such that every 2-dimensional linear subspace of W intersects the positive wedge K in a wedge of type $C_2^{(4)}$, then K must be a half-space. This is true for the wedge K with core $K\neq\varnothing$; but for infinite-dimensional wedge it is not true in general because the positive wedge corresponding to the Z-A Dictionary Ordering is a well-known counterexample (see [3], p. 18).

$P(i)$ is true for all $i<n$. Let $K'=T \cap K$. Then K' is not closed and the dimension of K' is less than that of K. We show that K' does not have the HBEP. In fact, if K' is not a wedge satisfying the condition of Case (2iii), then by the established part of this theorem K' does not have the HBEP. Also, if K' is a wedge satisfying the condition of Case (2iii), i.e., $K' \cap T'$ is not closed where $T'=\{v \in T \mid v, -v \in \bar{K}'\} \neq \{0\}$, then by the induction assumption K' does not have the HBEP. Now by the result from the original proof of this case we know that if K' does not have the HBEP then K does not have the HBEP. Thus, K does not have the HBEP. This proves that if $P(i)$ is true for all $i<n$ then $P(n)$ is true. Therefore we have proved K does not have the HBEP in Case (iii). This contradiction establishes the proof.

Bibliography

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